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# A subfamily of complex error functions

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## 1 Introduction

A complex error function is a transcendental entire function given by the form

$$C_{a,b}(z)=a\int_{0}^{z}e^{-w^{2}}dw+b$$

with $a\in \mathbb{C}\setminus\{0\}$ and $b\in \mathbb{C}$. It has two asymptotic values $\pm a\sqrt{\pi}/2+b$ and has no other singular value. In [3], a subfamily of complex error functions given by the form

$$C_{a,\sqrt{B}}(z)=a\int_{0}^{z}e^{-w^{2}}dw+\sqrt{B}$$

with $a\in \mathbb{R}\setminus\{0\}$ and $B\in \mathbb{R}$ is considered. Hence the family is described by two real parameters. Fatou components of some functions of this family have common boundary curves. In this note, we consider a subfamily of complex error functions given by the form

$$f_{a}(z)=a\int_{0}^{z}e^{-w^{2}}dw$$

with $a\in \mathbb{C}\setminus\{0\}$. Hence the family is described by one holomorphic parameter. A well-known family of transcendental entire functions with one complex parameter is an exponential family. It is studied by Baker and Rippon [1], Devaney [2] and others.

## 2 Results

We say that $f_{a}$ is hyperbolic if the orbit of each asymptotic value accumulates to attracting cyclic points. A connected component of the set of parameters

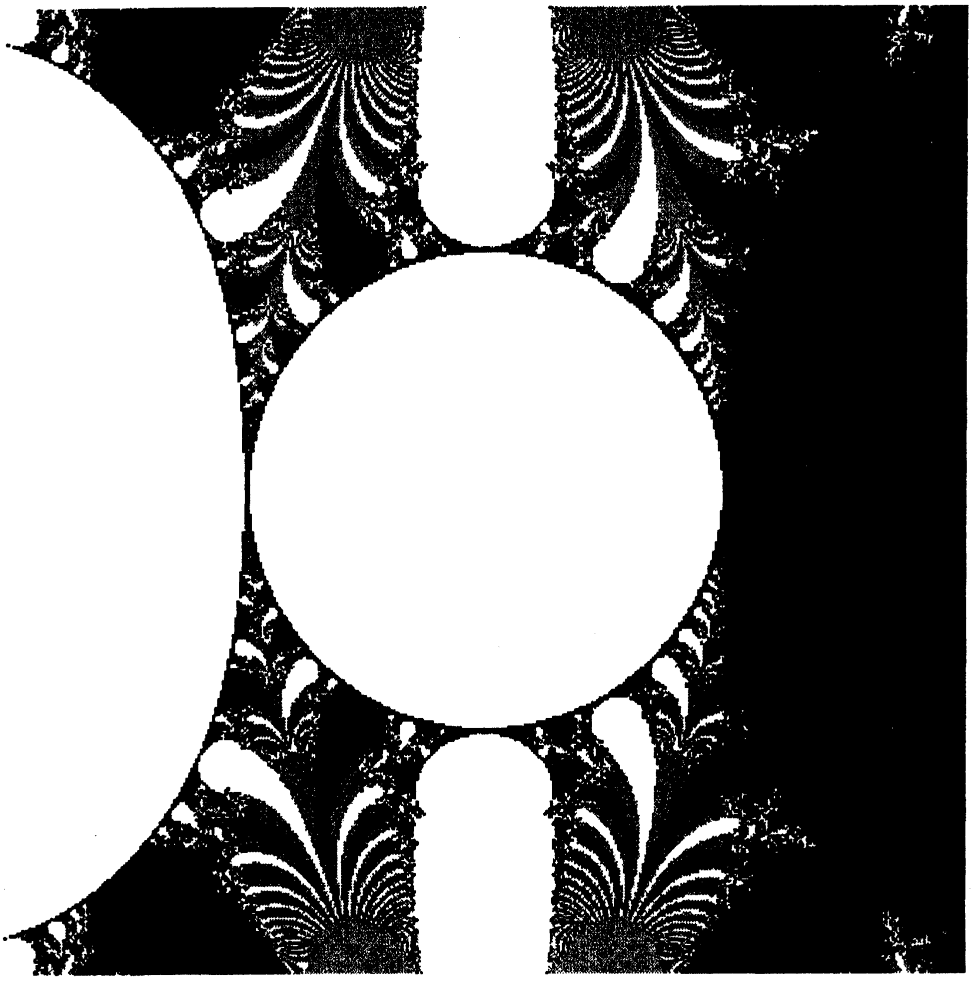

Figure 1: The parameter space of $f_a(z)$. The range shown is $|\Re a| \leq 2$ , $|\Im a| \leq 2$. The disk in the center is $A$. Hyperbolic components of $B_n$ are colored white and those of $D_n$ are colored black.

for which $f_a$ is hyperbolic is called a hyperbolic component. It is known that hyperbolic components are open.

We define subsets in the parameter space of $f_a$ as follows:

$$\begin{aligned} A &= \{a \mid f_a \text{ has a completely invariant component.}\}, \\ B_n &= \{a \mid f_a \text{ has only one attracting cycle with the period } 2n.\}, \\ D_n &= \{a \mid f_a \text{ has two attracting cycles with the period } n.\}, \end{aligned}$$

for $n \in \mathbb{N}$.

If there exists a cycle $\{z_1, z_2, \cdots, z_n\}$, then $\{-z_1, -z_2, \cdots, -z_n\}$ is also a cycle from the equation

$$f_a(-z) = -f_a(z).$$

Furthermore, we see that if the cycle is attracting, repelling or indifferent, then so is the corresponding one, respectively. The Maclaurin expansion of

$Er(z) = f_1(z)$ is of the form

$$Er(z) = \int_0^z e^{-w^2} dw = z - \frac{z^3}{3} + \cdots .$$

Adding further investigation on properties of $Er(z)$, we have the following theorem.

**Theorem 1.** *Every hyperbolic component is contained in one of $A$, $B_n$ and $D_n$. Furthermore, $A$ is also described by $\{a \mid 0 < |a| < 1\}$. Each of $B_1$ and $D_1$ consists of only one component.*

By the arguments similar to those in [1], we have the following theorems.

**Theorem 2.** *Every hyperbolic component except $A$ is simply-connected and unbounded.*

**Theorem 3.** *Each of $B_n$ and $D_n$ contains a component which is tangent to $A$.*

Cyclic Fatou components of the function belonging to a hyperbolic component tangent to $A$ attach to each other at the origin. By the arguments similar to those in [3], we have the following theorem.

**Theorem 4.** *Fatou components of $f_a$ belonging to a hyperbolic component tangent to $A$ have common boundary curves.*

# References


[1] I. N. Baker and P. J. Rippon, Iteration of exponential functions, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **9**(1984), 47–77.

[2] R. L. Devaney, Complex dynamics and entire functions, in *Complex Dynamical Systems, Proceeding of Symposia in Applied mathematics 49* (American Mathematical Society, Providence, 1994), 181–206.

[3] S. Morosawa, Fatou components whose boundaries have a common curve, Fund. Math. 183(2004), 47–57.

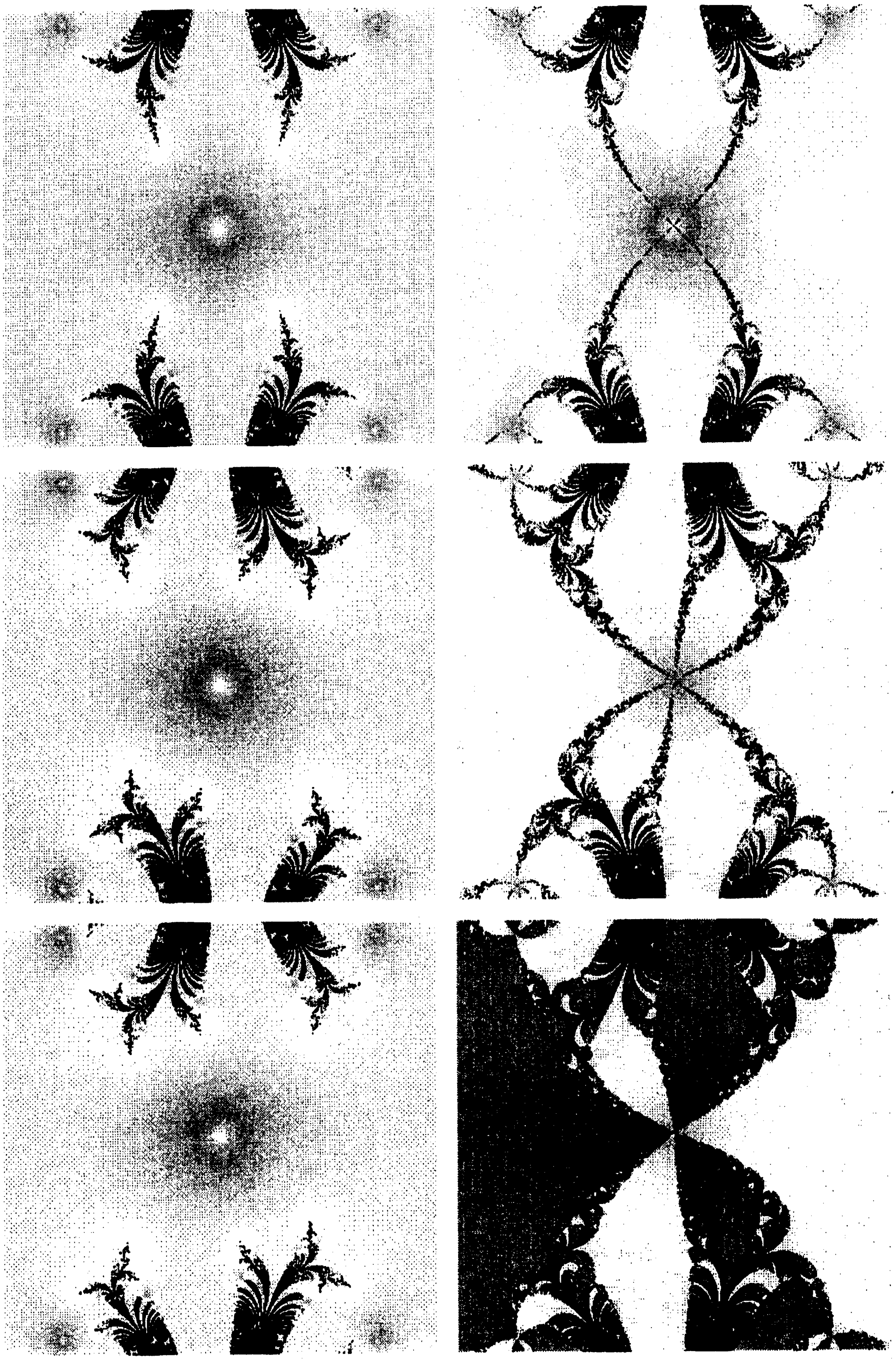

Figure 2: The Julia sets of $f_a(z)$. The range shown is $|\Re z| \leq 2$ , $|\Im z| \leq 2$. Upper left: $a = 0.95i$. Upper right: $a = 1.05i$. Middle left: $a = 0.475 + 0.8227241i$. Middle right: $a = 0.55 + 0.952628i$. Lower left: $a = -0.475 + 0.8227241i$. Lower right: $a = -0.55 + 0.952628i$.